Last time:

$$\left[\frac{\partial}{\partial x^\mu}\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)\right]_{anom}$$

$$= \frac{1}{(2\pi)^{12}} D_{\alpha\beta\gamma}\int d^4k_1\, d^4k_2\, e^{-i(k_1+k_2)\cdot x}\, e^{ik_1\cdot x}\, e^{ik_2\cdot z}$$

$$\times\, 4\pi^2 \epsilon^{\kappa\nu\lambda\rho} k_{1\kappa} k_{2\lambda}$$

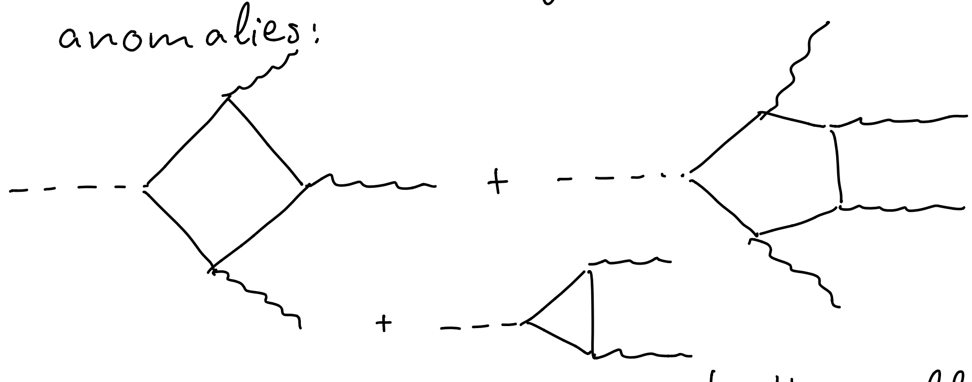
$$= -\frac{1}{4\pi^2} D_{\alpha\beta\gamma}\, \epsilon^{\kappa\nu\lambda\rho}\, \frac{\partial\delta^4(y-x)}{\partial y^\kappa}\, \frac{\partial\delta^4(z-x)}{\partial z^\lambda} \qquad (1)$$

$\rightarrow$ In the presence of gauge fields, we get the contribution:

$$\langle J^\mu_\alpha(x)\rangle_\Delta = -\frac{1}{2}\int d^4y\, d^4z\, \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)\, A^\beta_\nu(y)\, A^\gamma_\rho(z)$$

$$\rightarrow \left[\langle \partial_\mu J^\mu_\alpha(x)\rangle_\Delta\right]_{anom} = -\frac{1}{8\pi^2} D_{\alpha\beta\gamma}\, \epsilon^{\kappa\nu\lambda\rho}\, \partial_\kappa A^\beta_\nu(x)\, \partial_\lambda A^\gamma_\rho(x)$$

There are additional diagrams which contribute to anomalies:



Gauge invariance requires that they all add up to

$$\left[\langle \partial_\mu J^\mu_\alpha(x)\rangle_\Delta\right]_{anom} = -\frac{1}{32\pi^2} D_{\alpha\beta\gamma}\, \epsilon^{\kappa\nu\lambda\rho}\, F^\beta_{\kappa\nu}(x)\, F^\gamma_{\lambda\rho}(x) \qquad (2)$$

Let's now consider Fermion masses:

$$\mathcal{L}_{mass} = -\sum_{nn'\sigma\sigma'} \chi_{\sigma n}\, \varepsilon_{\sigma\sigma'}\, M_{nn'}\, \chi_{\sigma' n'} + \text{h.c.} \quad (3)$$

where σ is the two-component spinor index of the $(\frac{1}{2}, 0)$ rep. of Lorentz group, $\varepsilon_{\sigma\sigma'}$ is anti-sym. matrix with $\varepsilon_{\frac{1}{2},-\frac{1}{2}} = +1$, and

$$M = \frac{1}{2}\begin{pmatrix} 0 & m \\ m^T & 0 \end{pmatrix}$$

is mass matrix. M must satisfy

$$-T_\alpha^T M = M T_\alpha \quad (4)$$

Using $(T_\alpha)_{rs,r's'} = \delta_{rr'} (T_\alpha^{(r)})_{s,s'}$, ($r$ denotes irreducible rep.)

$$M_{rs,r's'} = (M^{(rr')})_{ss'},$$

eq. (4) becomes

$$-T_\alpha^{(r)T} M^{(r,r')} = M^{(r,r')} T_\alpha^{(r')}$$

Schur's lemma $\longrightarrow$ $-T_\alpha^{(r)T}$ and $T_\alpha^{(r')}$ are related by similarity trf.

Then $D_{\alpha\beta\gamma} = \frac{1}{2} \mathrm{tr}\left[\{T_\alpha, T_\beta\} T_\gamma\right]$ gives

$$D^{(r)}_{\alpha\beta\gamma} = -D^{(r')}_{\alpha\beta\gamma}$$

→ so the anomaly either vanishes or cancels between the two reps. (for $r \neq r'$)

→ anomalies in a given set of symmetries are unaffected by possible presence of fermions with a mass allowed by these symmetries.

Let us now combine left-handed and right-handed fields into a single spinor Ψ with Lagrangian $-\bar{\Psi} \not{D} \Psi$, where

$$\not{D} = \not{\partial} - i\not{A}_\alpha T_\alpha \left(\frac{1+\gamma_5}{2} \right)$$

→ one-loop vacuum functional is $\operatorname{Det} \not{D}$

→ failure of gauge invariance comes from splitting

$$\not{D} = \underbrace{\not{\partial}\left(\frac{1-\gamma_5}{2}\right)}_{\text{non-gauge inv.}} + \underbrace{\not{D}\left(\frac{1+\gamma_5}{2}\right)}_{\text{gauge inv.}}$$

§6.3 Anomaly-Free Gauge Theories

Anomaly is proportional to $D_{\alpha\beta\gamma}$

$\longrightarrow$ for gauge currents we must have

$$D_{\alpha\beta\gamma} \equiv \frac{1}{2} tr\left[\{T_\alpha, T_\beta\} T_\gamma\right] = 0, \qquad (*)$$

where T_α is rep. of gauge algebra on the set of all left-handed fermions

$(*)$ is satisfied when

$$(iT_\alpha)^* = S(iT_\alpha)S^{-1}$$

or equivalently (since T_α are hermitian)

$$T_\alpha^T = -S T_\alpha S^{-1}$$

$\longrightarrow$ inserting into $D_{\alpha\beta\gamma} = \frac{1}{2} tr\left[\{T_\alpha, T_\beta\} T_\gamma\right]$

gives $D_{\alpha\beta\gamma} = -D_{\alpha\beta\gamma}$

$\longrightarrow$ rep. is either "real" or "pseudoreal"

(real means here $\exists$ similarity trf. with

$$\underset{\substack{\uparrow \\ \text{anti-sym.}}}{iA} = T_\alpha' = R T_\alpha R^{-1}$$

)

$\longrightarrow$ no anomaly for gauge algebras which are real or pseudo-real examples:

- $SO(2n+1)$ (including $SU(2) \cong SO(3)$)
- $SO(4n)$ for $n \geq 2$
- $USp(2n)$ for $n \geq 3$
- G_2
- F_4
- E_7 and E_8
- all directs sums of the above

Other algebras for which $D_{\alpha\beta\gamma} = 0$ (but neither real nor pseudoreal):

- $SO(4n+2)$
- E_6

→ anomalies are only possible for gauge algebras which include $SU(n)$ ($n \geq 3$) or $U(1)$ factors

→ important for Standard model: $SU(3) \times SU(2) \times U(1)$

→ must rely on cancellations between quarks and leptons to get an anomaly free theory

1st generation of standard model:

Fermions	SU(3)	SU(2)	U(1) $[Y/g']$
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	$-1/6$
u_R^*	$\bar{3}$	1	$+2/3$
d_R^*	$\bar{3}$	1	$-1/3$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2	$1/2$
e_R^*	1	1	-1

Since generators of SU(n) are trace-less for any representation, we only need to consider invariant combinations of T_α, T_β and T_γ under $SU(3) \times SU(2)$:

$$SU(3) \times SU(3) = 1 + \cdots$$
$$SU(3) \times SU(3) \times SU(3) = 1 + \cdots$$
$$SU(2) \times SU(2) = 1 + \cdots$$
$$SU(2) \times SU(2) \times SU(2) = 1 + \cdots$$

- $SU(3) - SU(3) - SU(3)$:
  left-handed fermions trf. as $3 + 3 + \bar{3} + \bar{3} + 1 + 1 + 1$ of SU(3) $\longrightarrow$ real rep. $\longrightarrow D_{\alpha\beta\gamma} = 0$

- $SU(3) - SU(3) - U(1)$:

$$\text{anomaly} \propto \sum_{3,\bar{3}} Y/g' = -\frac{1}{6} - \frac{1}{6} + \frac{2}{3} - \frac{1}{6} = 0$$

- $SU(2) - SU(2) - SU(2)$:

  $SU(2)$ has only real or pseudoreal reps.
  $\rightarrow$ no anomaly

- $SU(2) - SU(2) - U(1)$:

$$\text{anomaly} \propto \sum_{\text{doublets}} Y/g' = 3\left(-\frac{1}{6}\right) + \frac{1}{2} = 0$$

- $U(1) - U(1) - U(1)$:

$$\sum (Y/g')^3 = 6\left(-\frac{1}{6}\right)^3 + 3\left(\frac{2}{3}\right)^3 + 3\left(-\frac{1}{3}\right)^3 + 2\left(\frac{1}{2}\right)^3 + (-1)^3$$
$$= 0$$

Thus <u>all</u> anomalies cancel for the gauge symmetries of Standard Model.